How to simulate turbulent dynamos with the Pencil Code

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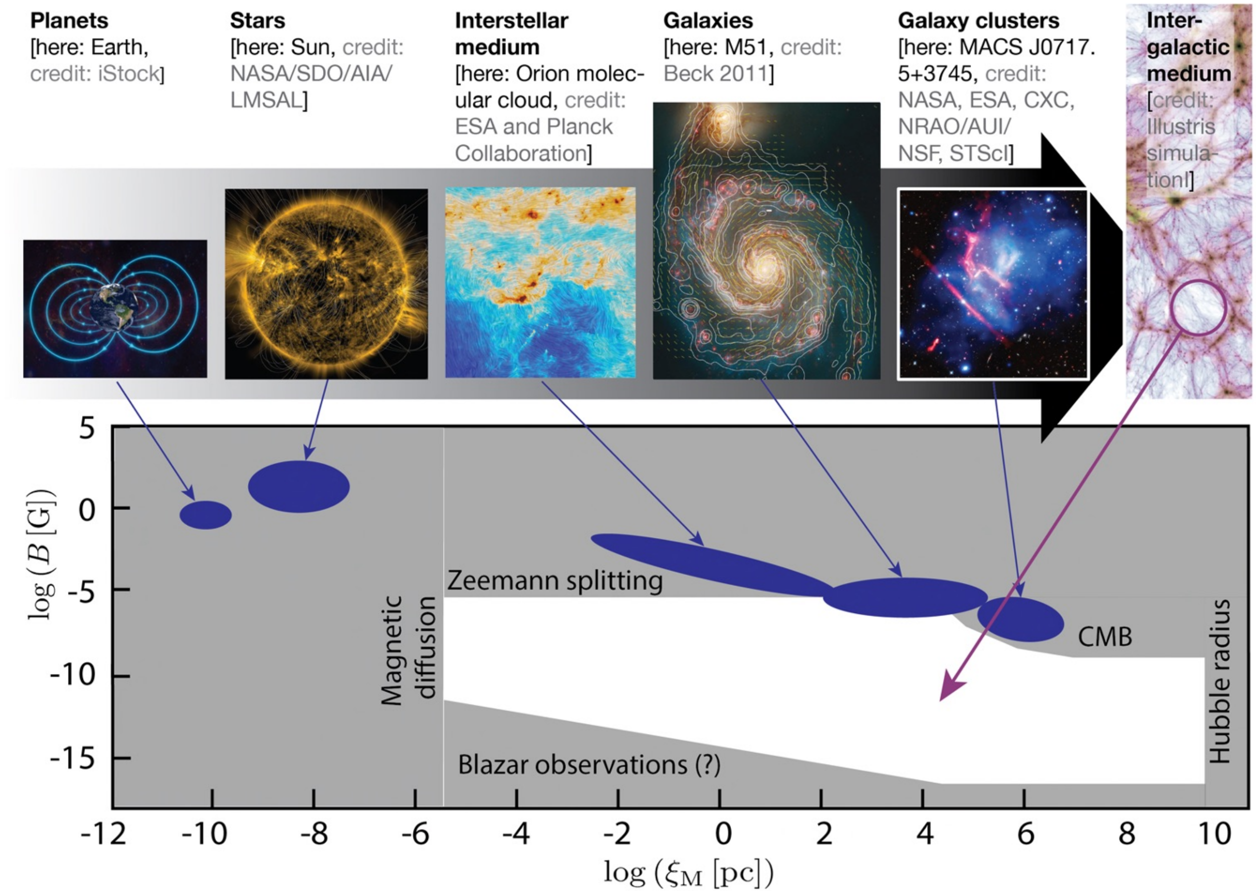
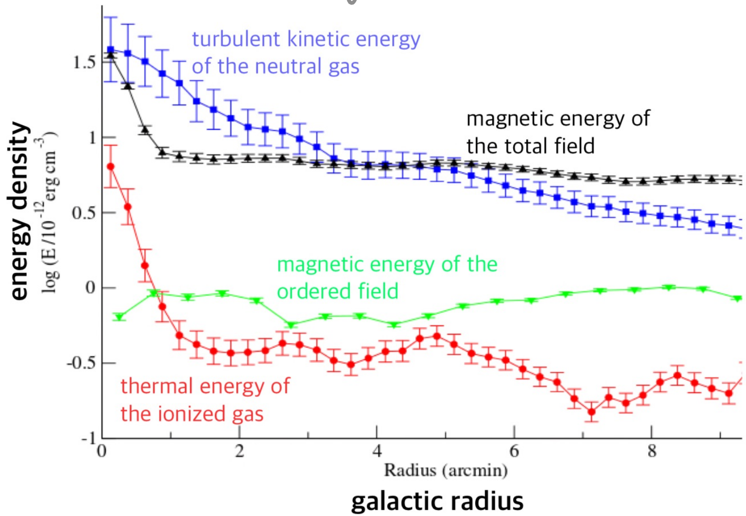
1. Cosmic magnetic fields & the need for dynamos

Strength and length scales



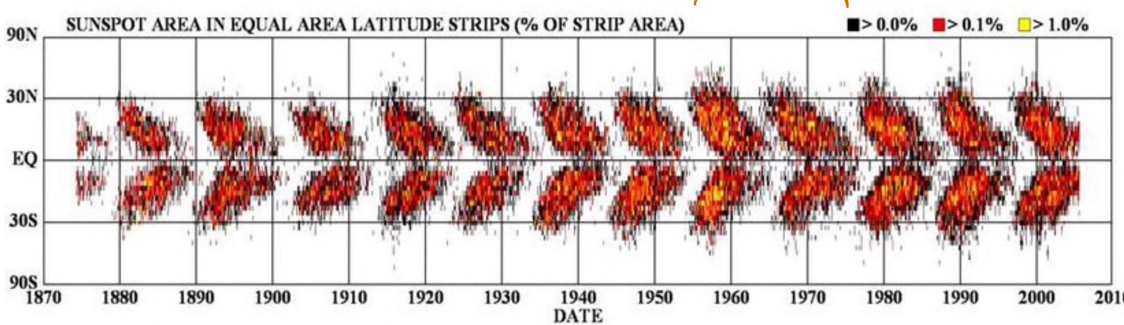
Structure (of galactic magnetic fields)



Magnetic energy density can be comparable to other characteristic energy densities.

Time evolution (in the Sun)

22 yrs



Magnetic fields can change periodically.

"Butterfly diagram"

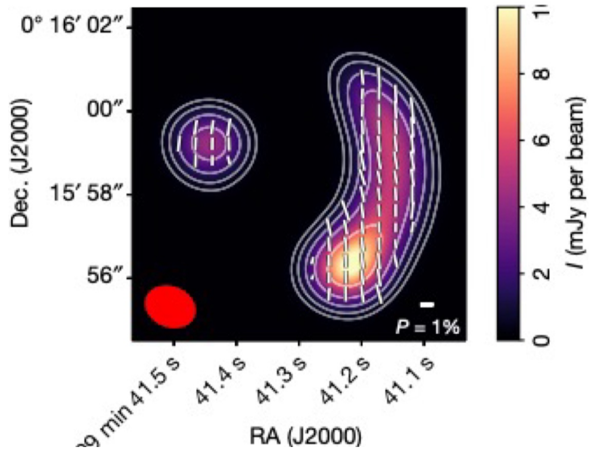
Dynamos in the context of cosmic history

Seed magnetic fields

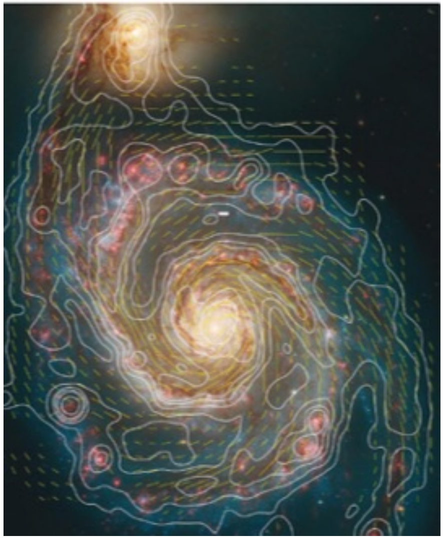
primordial

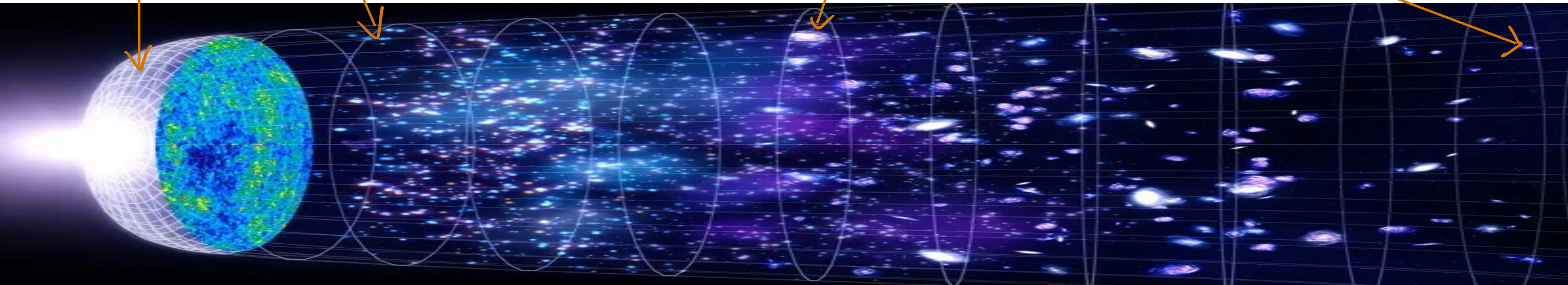
astrophysical

Magnetic field in a galaxy at $z = 2.6$
[Geach et al. 2023]



Magnetic field in a modern galaxy





2. Modelling dynamos

2.1 MHD equations

Mass conservation:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0}$$

Momentum conservation:

$$\boxed{\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla \left(p + \frac{B^2}{8\pi}\right) + \frac{1}{4\pi\rho} (\vec{B} \cdot \nabla) \vec{B} + \frac{1}{m} \vec{F} + \nu \nabla^2 \vec{v}}$$

Induction equation:

$$\boxed{\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) - \eta \nabla^2 \vec{B}}$$

Equation of state:

$$\boxed{\rho = \rho(p)}$$

<u>Limitations:</u>

- not applicable for high frequency phenomena (that involve charge separation) $\rightarrow$ $\omega \ll \omega_p$
- non-relativistic ($v^2/c^2 \ll 1$)
- assume transport coefficients are scalars $\rightarrow$ $\nu_c \gg \frac{\omega_g}{2\pi}$ (otherwise strong B-fields lead to anisotropies)

2.2 Dynamo definitions and classes

$$(\nabla \times \vec{a}) \cdot \vec{b} = \nabla \cdot (\vec{a} \times \vec{b}) + a \cdot (\nabla \times \vec{b})$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a})$$

Governing equations & energetics (incompressible fluid):

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) - \eta \nabla \times (\nabla \times \vec{B}) \qquad (*)$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla p + \frac{1}{4\pi\rho} (\nabla \times \vec{B}) \times \vec{B} + \frac{1}{m} \vec{F} + \nu \nabla^2 \vec{v} \qquad (**)$$

Multiply $(*)$ by $\frac{\vec{B}}{4\pi}$ and integrate over volume (and ignore fluxes through surfaces):

$$\int \frac{\partial \vec{B}}{\partial t} \cdot \frac{\vec{B}}{4\pi} dV = \int [\nabla \times (\vec{v} \times \vec{B})] \cdot \frac{\vec{B}}{4\pi} dV - \eta \int [\nabla \times (\nabla \times \vec{B})] \cdot \frac{\vec{B}}{4\pi} dV$$

$$\Rightarrow \underbrace{\frac{d}{dt} \int \left(\frac{\vec{B}^2}{8\pi}\right) dV}_{\text{magnetic energy}} = \int \frac{1}{4\pi} (\vec{v} \times \vec{B}) (\nabla \times \vec{B}) dV - \frac{\eta}{4\pi} \int (\nabla \times \vec{B})^2 dV \qquad \vec{j} = \frac{4\pi}{c} \nabla \times \vec{B}$$

$$= \frac{1}{c} \int (\vec{v} \times \vec{B}) \cdot \vec{j} \, dV - \frac{4\pi\eta}{c^2} \int \vec{j}^2 dV$$

$$= \underbrace{-\frac{1}{c} \int \vec{v} \cdot (\vec{j} \times \vec{B}) dV}_{\text{work of Lorentz force}} - \underbrace{\frac{1}{\sigma} \int \vec{j}^2 dV}_{\text{resistive losses}}$$

Multiply (xx) by $\rho\vec{v}$ and integrate over volume (and ignore fluxes through surfaces):

$$\underbrace{\frac{d}{dt}\int \frac{1}{2}\rho\vec{v}^2\,dV}_{\text{kinetic energy}} = \underbrace{+\frac{1}{c}\int \vec{v}\cdot(\vec{j}\times\vec{B})\,dV}_{\text{work of Lorentz force}} + \int \frac{\rho}{m}\vec{v}\cdot\vec{f}\,dV - 2\nu\int\rho(\nabla\times\vec{v})^2\,dV$$

$\Rightarrow$ "Definition" of a dynamo:

$$\boxed{\text{kinetic energy}} \xrightarrow[\text{= flow of energy}]{\text{MHD dynamo}} \boxed{\text{magnetic energy}}$$

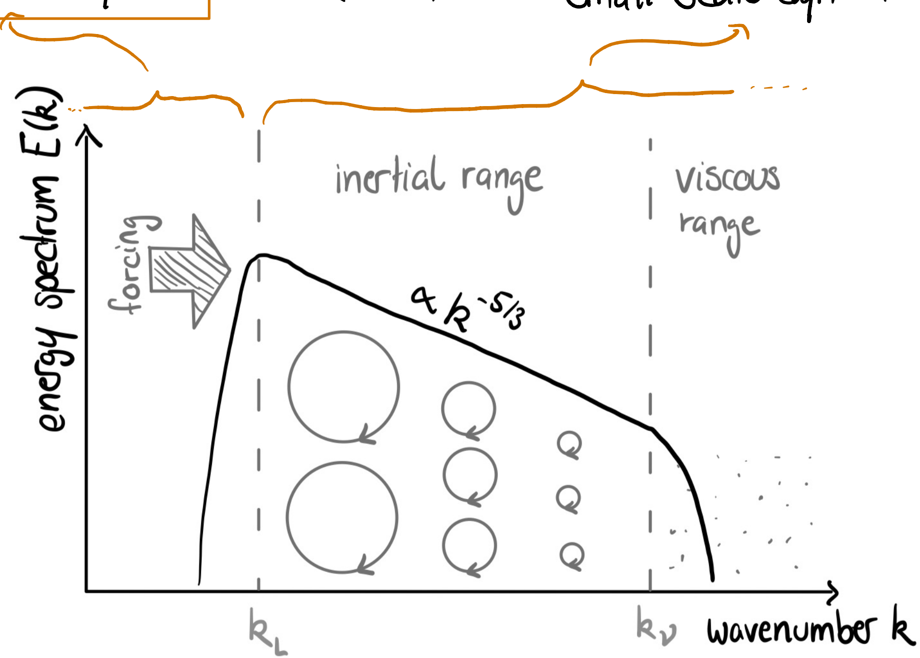
Different types of dynamos

- Kinematic dynamo ⟷ Nonlinear dynamo

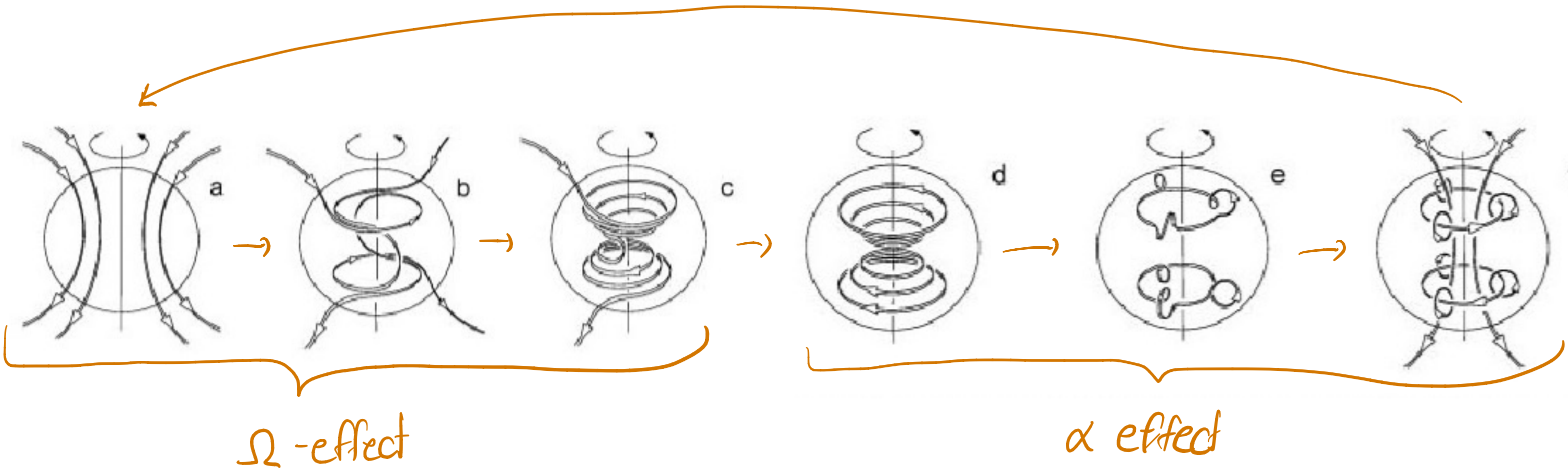
  $\vec{v}$ is given ⟹ need only (*)

  $\vec{v}$ is affected by $\vec{B}$ [via $\frac{1}{c}(\nabla\times\vec{B})\times\vec{B}$] ⟹ need (*) and (**)

- $\boxed{\text{Large-scale dynamo}}$ ⟷ Small-scale dynamo



Large-scale dynamo: phenomenology



2.3 Mean-field magnetohydrodynamics

→ Steenbeck, Krause & Rädler (1966)

Mean-field induction equation

Separation into mean fields and fluctuations:

$$\vec{B} \to \langle \vec{B} \rangle + \vec{B}'$$
$$\vec{v} \to \langle \vec{v} \rangle + \vec{v}'$$

$[\langle\langle \vec{B} \rangle\rangle = \langle \vec{B} \rangle$ and $\langle \vec{B}' \rangle = 0]$

Insert in induction equation:

$$\frac{\partial \langle \vec{B} \rangle}{\partial t} + \frac{\partial \vec{B}'}{\partial t} = \nabla \times \left[(\langle \vec{v} \rangle + \vec{v}') \times (\langle \vec{B} \rangle + \vec{B}') \right] + \eta \nabla^2 (\langle \vec{B} \rangle + \vec{B}') \quad (***)$$

| average

$$\Rightarrow \frac{\partial \langle \vec{B} \rangle}{\partial t} = \nabla \times \Big[\underbrace{\langle \langle \vec{v} \rangle \times \langle \vec{B} \rangle \rangle}_{\langle \vec{v} \rangle \times \langle \vec{B} \rangle} + \underbrace{\langle \langle \vec{v} \rangle \times \vec{B}' \rangle}_{0} + \underbrace{\langle \vec{v}' \times \langle \vec{B} \rangle \rangle}_{0} + \underbrace{\langle \vec{v}' \times \vec{B}' \rangle}_{\neq 0} \Big] + \eta \nabla^2 \langle \vec{B} \rangle$$

"mean electromotive force (EMF)" $= \mathcal{E}$

$$\Rightarrow \boxed{\frac{\partial \langle \vec{B} \rangle}{\partial t} = \nabla \times (\langle \vec{v} \rangle \times \langle \vec{B} \rangle) + \nabla \times \mathcal{E} + \eta \nabla^2 \langle \vec{B} \rangle} \quad (****)$$

What is $\mathcal{E}$?

If $\vec{v}'$ and $\vec{B}'$ are uncorrelated : $\mathcal{E} = \langle \vec{v}' \times \vec{B}' \rangle = \langle \vec{v}' \rangle \times \langle \vec{B}' \rangle = 0$

$\Rightarrow$ Is there some correlation ? Yes, because $\vec{B}'$ is caused by $\vec{v}'$.

i) Find equation for $\vec{B}'$

Subtract (****) from (***):

$$\frac{\partial \vec{B}'}{\partial t} = \nabla \times \left[\langle \vec{v} \rangle \times \vec{B}' + \vec{v}' \times \langle \vec{B} \rangle + \vec{v}' \times \vec{B}' - \mathcal{E} \right] + \eta \nabla^2 \vec{B}'$$

ii) First-order smoothing approximation : $\vec{B}'$ remains small during correlation time τ

$\Rightarrow$ ignore all terms linear in $\vec{B}'$ (incl. $\mathcal{E}$)

$$\Rightarrow \frac{\vec{B}'}{\tau} \approx \nabla \times \left(\vec{v}' \times \langle \vec{B} \rangle \right)$$

$$\Leftrightarrow \vec{B}' \approx \tau \left(\langle \vec{B} \rangle \cdot \nabla \right) \vec{v}' - \tau \left(\vec{v}' \cdot \nabla \right) \langle \vec{B} \rangle$$

Use to evaluate EMF:

$$\mathcal{E}_i = \langle \vec{v}' \times \vec{B}' \rangle_i$$

$$= \langle \varepsilon_{ijk} v_j' B_k' \rangle \qquad \text{| insert } \vec{B}'$$

$$= \left\langle \varepsilon_{ijk} v_j' \tau \langle B \rangle_l \frac{\partial v_k'}{\partial x_l} \right\rangle - \left\langle \varepsilon_{ijk} \tau v_j' v_l' \frac{\partial \langle B \rangle_k}{\partial x_l} \right\rangle$$

$$= \underbrace{\varepsilon_{ijk} \left\langle v_j' \frac{\partial v_k'}{\partial x_l} \right\rangle \tau}_{\equiv \alpha_{il}} \langle B \rangle_l - \underbrace{\varepsilon_{ijk} \langle v_j' v_l' \rangle \tau}_{\equiv -\beta_{ikl}} \frac{\partial \langle B \rangle_k}{\partial x_l}$$

depend only on statistical properties of velocity field

iii) Assume isotropic turbulence

$$\alpha_{il} = \alpha\, \delta_{il} \quad \Rightarrow \quad \boxed{\alpha = -\frac{1}{3}\langle \vec{v}' \cdot (\nabla \times \vec{v}') \rangle \tau}$$

$$\beta_{ikl} = -\eta_T\, \varepsilon_{ikl} \quad \Rightarrow \quad \boxed{\eta_T = \frac{1}{3}\langle \vec{v}' \cdot \vec{v}' \rangle \tau}$$

$$\Rightarrow \boxed{\frac{\partial \langle \vec{B} \rangle}{\partial t} = \nabla \times (\langle \vec{v} \rangle \times \langle \vec{B} \rangle) + \nabla \times (\alpha \langle \vec{B} \rangle) + (\eta + \eta_T) \nabla^2 \langle \vec{B} \rangle}$$

Simple dynamo solutions

- α^2 dynamo : $\langle \vec{v} \rangle = 0$

$$\Rightarrow \frac{\partial \langle \vec{B} \rangle}{\partial t} = \nabla \times (\alpha \langle \vec{B} \rangle) + (\eta + \eta_T) \nabla^2 \langle \vec{B} \rangle$$

Ansatz : $\langle \vec{B} \rangle (x) = \hat{\vec{B}}(\vec{k}) \exp(i\vec{k}\vec{x} + \gamma t)$

$$\Rightarrow \gamma \hat{\vec{B}} = \alpha i \vec{k} \times \hat{\vec{B}} - (\eta + \eta_T) k^2 \hat{\vec{B}}$$

$$\Rightarrow \gamma \hat{\vec{B}} = \begin{pmatrix} -(\eta+\eta_T)k^2 & -i\alpha k_z & i\alpha k_y \\ i\alpha k_z & -(\eta+\eta_T)k^2 & -i\alpha k_x \\ -i\alpha k_y & i\alpha k_x & -(\eta+\eta_T)k^2 \end{pmatrix} \hat{\vec{B}}$$

$$\Rightarrow \left(\gamma + (\eta+\eta_T)k^2\right)\left[\left(\gamma + (\eta+\eta_T)k^2\right)^2 - \alpha^2 k^2\right] = 0$$

$$\Rightarrow \gamma_0 = -(\eta+\eta_T)k^2 \quad \& \quad \boxed{\gamma_\pm = \pm|\alpha k| - (\eta+\eta_T)k^2}$$

$$\frac{d\gamma_+}{dk} = -2(\eta+\eta_T)k + |\alpha| \overset{!}{=} 0$$

$$\Rightarrow \boxed{k_{max} = \frac{|\alpha|}{2(\eta+\eta_T)}}$$

$$\gamma_{max} = \left|\frac{\alpha^2}{2(\eta+\eta_T)}\right| - \frac{\alpha^2}{4(\eta+\eta_T)}$$

$$\Rightarrow \boxed{\gamma_{max} = \frac{\alpha^2}{4(\eta+\eta_T)}}$$

• $\alpha\Omega$-dynamo: $\langle \vec{v} \rangle = (0, S\cdot x, 0)$

shear, e.g. $S = -\frac{3}{2}\Omega$ for Keplerian disc

$S = r\frac{\partial \Omega}{\partial r}$ in Sun

Growth rates:

$\Rightarrow \gamma_0 = -(\eta + \eta_T)k^2$

$$\boxed{\gamma_\pm = \pm\left|\frac{1}{2}\alpha S k_z\right|^{1/2} - (\eta + \eta_T)k^2}$$

for axisymmetric solutions $k_y = 0$

Oscillations:

$$\boxed{\omega_\pm = \left|\frac{1}{2}\alpha S k_z\right|^{1/2}}$$

3. Numerical methods in hydrodynamics

3.1 Basic concepts

Fluid dynamics → set of coupled Partial Differential Equations (PDEs)

→ need numerical tools

Hydrodynamical equations in conservative form

Most convenient form of equations:

$$\boxed{\frac{\partial}{\partial t}(\text{density of quantity}) + \nabla \cdot (\text{its flux}) = \text{sources/sinks}}$$

Significance of this form: Without sources/sinks, the value of a quantity can only change if there are fluxes through the boundary surface.

Dynamical equations for ideal fluids :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

$$\frac{\partial (\rho \vec{v})}{\partial t} + \nabla \cdot (\rho \vec{v} \otimes \vec{v} + p \mathbb{1}) = 0$$

$$\Big| \; \nabla \cdot (\vec{v} \otimes \vec{v}) = \underbrace{(\nabla \cdot \vec{v})}_{=0} \cdot \vec{v} + (\vec{v} \cdot \nabla) \vec{v}$$

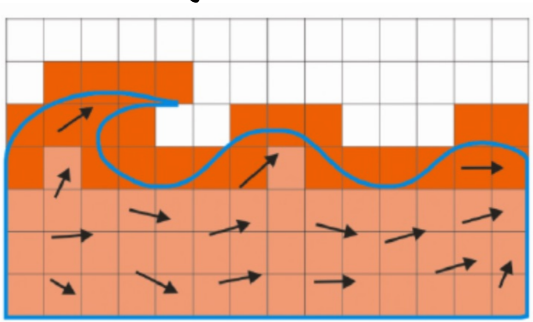
$$\frac{\partial}{\partial t}\left(\rho \varepsilon + \frac{1}{2} \rho v^2\right) + \nabla \cdot \left[\left(\frac{1}{2} v^2 + \varepsilon + \frac{p}{\rho}\right) \rho \vec{v}\right] = 0$$

or, in compact form :

$$\boxed{\frac{\partial}{\partial t} \vec{q} + \nabla \cdot \mathcal{F}(\vec{q}) = 0 \quad \text{with } \vec{q} \equiv \begin{pmatrix} \rho \\ \rho \vec{v} \\ \rho \varepsilon + \frac{1}{2} \rho v^2 \end{pmatrix} \text{ and } \mathcal{F}(\vec{q}) \equiv \begin{pmatrix} \rho \vec{v} \\ \rho \vec{v} \otimes \vec{v} + p \mathbb{1} \\ \left(\frac{1}{2} v^2 + \varepsilon + \frac{p}{\rho}\right) \rho \vec{v} \end{pmatrix}} \quad (*)$$

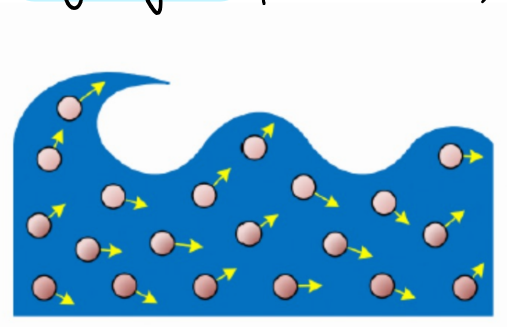
Discretization: Eulerian vs. Lagrangian methods

Eulerian (grid-based)



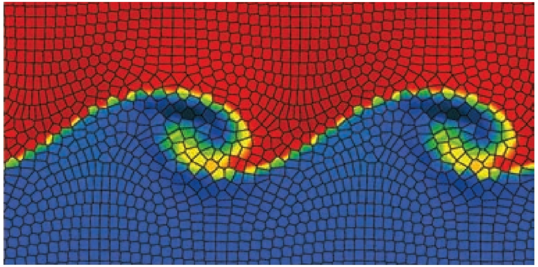
- solving (*) at fixed positions
- evolution of $\vec{q}$ is described by Eulerian derivative
$$\frac{\partial \vec{q}}{\partial t}$$
- "Adaptive Mesh Refinement" (AMR) possible

Lagrangian (particle-based)



- solving (*) comoving with the fluid
- evolution of $\vec{q}$ is described by Lagrangian derivative
$$\frac{d\vec{q}}{dt} = \frac{\partial \vec{q}}{\partial t} + (\vec{v} \cdot \nabla)\vec{q}$$
- "Smoothed-Particle Hydrodynamics" (SPH)

Mixture:



- moving mesh

3.2 Finite difference methods for PDEs

Strategy

1. Define a spatial grid, $\boldsymbol{x}_i$, where the index i refers to one of the N grid points.

2. Specify initial conditions for $\boldsymbol{q}(\boldsymbol{x}, t = 0)$ at all $\boldsymbol{x}_i$, and suitable boundary conditions on the finite computational domain.

3. Compute $\boldsymbol{q}(\boldsymbol{x}_i, t = 0)$ at all $\boldsymbol{x}_i$.

4. Compute the fluxes $\mathcal{F}(\boldsymbol{q})$ at all $\boldsymbol{x}_i$.

5. Take a small time step Δt and calculate the new $\boldsymbol{q}(\boldsymbol{x}_i, t + \Delta t)$.

6. Go back to step 3.

x times

Hydrodynamical PDEs: analytical solutions

The equations (*) are hyperbolic: The Jacobian matrix of the flux,

$$F_{ij}(\vec{q}) \equiv \frac{\partial F_i}{\partial q_j},$$

has only real eigenvalues and is diagonizable.

1D example: Linear advection eq.

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} = 0 \quad \Big| \text{ insert formal solution } \rho(x,t) = \rho_0 + \rho_1 e^{i(kx-\omega t)}$$

$$\Rightarrow \omega = kv$$

$$\Rightarrow \rho(x,t) = \rho_0 + \rho_1 e^{ik(x-vt)}$$

$\Rightarrow$ ω is always real $\Rightarrow$ travelling wave with group velocity

$$v_{gr} = \frac{\partial \omega}{\partial k} = v$$

$\Rightarrow$ All modes travel at same speed v.

Transport phenomena are usually described by a very different kind of PDE, so-called parabolic PDEs.

<u>1D example</u>: Heat eq.

$$\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2}$$

| insert formal solution : $T(x,t) = T_0 + T_1 e^{i(kx - \omega t)}$

$\Rightarrow \omega = -iKk^2$

$\Rightarrow T(x,t) = T_0 + T_1 e^{ikx} e^{-Kk^2 t}$

$\Rightarrow$ exponential decay

$\Rightarrow$ Perturbation will die out.

Consider (*) in 1D:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \qquad (1D\,*)$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v^2 + \overset{0}{\overset{\shortparallel}{p}}) = 0 \qquad (1D\,**)$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho \varepsilon\right) + \frac{\partial}{\partial x}\left(\left[\frac{1}{2}\rho v^2 + \rho\varepsilon + p\right]v\right) = 0 \qquad (1D\,***)$$

Assume that p = const and v = const :

$$\xrightarrow[\text{and } (1D**)]{(1D*)} \quad \boxed{\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} = 0} \qquad (A*)$$

$$\xrightarrow{(1D***)} \quad \cancel{\frac{1}{2}v^2\frac{\partial \rho}{\partial t}} + \cancel{\rho\frac{\partial \varepsilon}{\partial t}} + \varepsilon\frac{\partial \rho}{\partial t} + \cancel{\frac{1}{2}v^3\frac{\partial \rho}{\partial x}} + \cancel{\rho\frac{\partial \varepsilon}{\partial x}v} + \varepsilon\frac{\partial \rho}{\partial x}v = 0$$

$$\Rightarrow \quad \boxed{\frac{\partial \varepsilon}{\partial t} + v\frac{\partial \varepsilon}{\partial x} = 0} \qquad (A**)$$

(A*) and (A**) are formally identical, known as the linear advection equation.

Solutions of (A*) and (A**), assuming initial conditions

$$\rho(x, t=0) = \rho_0(x)$$
$$\varepsilon(x, t=0) = \varepsilon_0(x)$$

are trivial:

$$\boxed{\begin{aligned} \rho(x,t) &= \rho_0(x - vt) \\ \varepsilon(x,t) &= \varepsilon_0(x - vt) \end{aligned}}$$

$\Rightarrow$ Use this to test numerical schemes.

Hyperbolic PDEs: Numerical solutions

Courant-Friedrichs-Lewy condition (short: Courant criterion)

Solutions of hyperbolic PDEs travel with finite speed v.

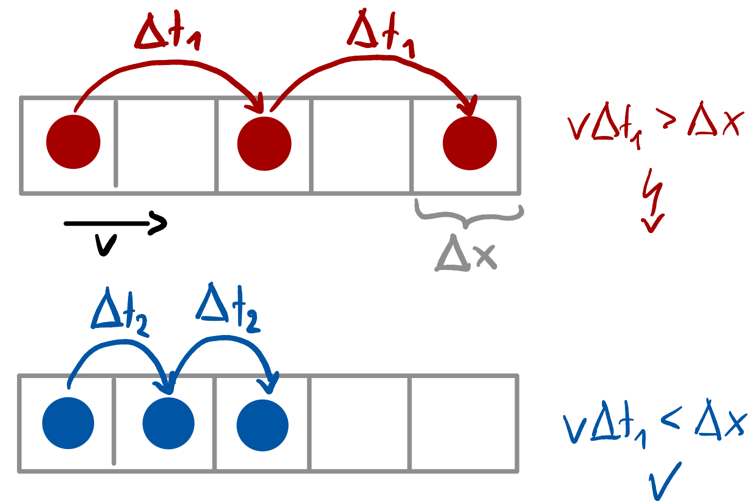
To capture the information in every grid we need:

$$|v \cdot \Delta t| < \Delta x .$$

Or:

$$\boxed{\left|\frac{v \cdot \Delta t}{\Delta x}\right| = \alpha_c \leq 1}.$$

$\uparrow$ Courant parameter



Finite difference method

Linear advection eq. for quantity q:

$$\frac{\partial q}{\partial t} + v \frac{\partial q}{\partial x} = 0$$

Discretization in space and time:

$$x_i = x_0 + i\Delta x$$

$$t^n = t_0 + n\Delta t$$

How can derivatives be expressed in terms of discretized quantities?

i) Taylor expansion:

$$q_{i-1} \equiv q(x_i - \Delta x)$$

$$= q(x_i) - \Delta x \frac{\partial q}{\partial x_i}(x_i) + \frac{(\Delta x)^2}{2} \frac{\partial^2 q}{\partial x^2}(x_i) - \frac{(\Delta x)^3}{6} \frac{\partial^3 q}{\partial x^3}(x_i) + \mathcal{O}(\Delta x^4)$$

$$q_i \equiv q(x_i)$$

$$q_{i+1} \equiv q(x_i + \Delta x)$$

$$= q(x_i) + \Delta x \frac{\partial q}{\partial x}(x_i) + \frac{(\Delta x)^2}{2} \frac{\partial^2 q}{\partial x^2}(x_i) + \frac{(\Delta x)^3}{6} \frac{\partial^3 q}{\partial x^3}(x_i) + \mathcal{O}(\Delta x^4)$$

ii) Finite difference approximation for spatial derivatives:

- 1st order derivative, lowest accuracy:
  - Forward difference: $\frac{\partial q}{\partial x}(x_i) \approx \frac{q_{i+1} - q_i}{\Delta x} - \frac{\Delta x}{2} \frac{\partial^2 q}{\partial x^2}(x_i)$
  - Backward difference: $\frac{\partial q}{\partial x}(x_i) \approx \frac{q_i - q_{i-1}}{\Delta x} + \frac{\Delta x}{2} \frac{\partial^2 q}{\partial x^2}(x_i)$

  } 1st order accurate

  - Centered difference: $\frac{\partial q}{\partial x}(x_i) \approx \frac{q_{i+1} - q_{i-1}}{2\Delta x} \underbrace{- \frac{(\Delta x)^2}{6} \frac{\partial^3 q}{\partial x^3}(x_i)}_{\text{truncation error}}$

  } 2nd order accurate

- 2nd derivative; e.g.:

$$\frac{\partial^2 q_i}{\partial x^2}(x_i) \approx \frac{q_{i+1} - 2q_i + q_{i-1}}{(\Delta x)^2} - \frac{1}{12} \frac{\partial^4 q}{\partial x^4}(x_i)(\Delta x)^2$$

- 1st derivative, higher accuracy, e.g.:

$$\frac{\partial q_i}{\partial x}(x_i) \approx \frac{-q_{i+2} + 8q_{i+1} - 8q_{i-1} + q_{i-2}}{12\Delta x} + \mathcal{O}(\Delta x^4)$$

iii) Finite difference approximation for time derivatives (only backward difference schemes):

$$\frac{\partial q_i}{\partial t} \approx \frac{q_i^{n+1} - q_i^{n}}{\Delta t}$$

iv) Use in evolution eq:

$$\frac{\partial q}{\partial t} + v \frac{\partial q}{\partial x} = 0$$

| E.g.: using central space scheme

$$\Rightarrow \frac{q_i^{n+1} - q_i^{n}}{\Delta t} + v \frac{q_{i+1}^{n} - q_{i-1}^{n}}{2\Delta x} = 0$$

$$\Rightarrow \boxed{q_i^{n+1} = q_i^{n} - \Delta t\, v \frac{q_{i+1}^{n} - q_{i-1}^{n}}{2\Delta x}}$$

"Forward Time Central Space (FTCS) scheme"

"Explicite Euler Scheme"

Different Euler schemes

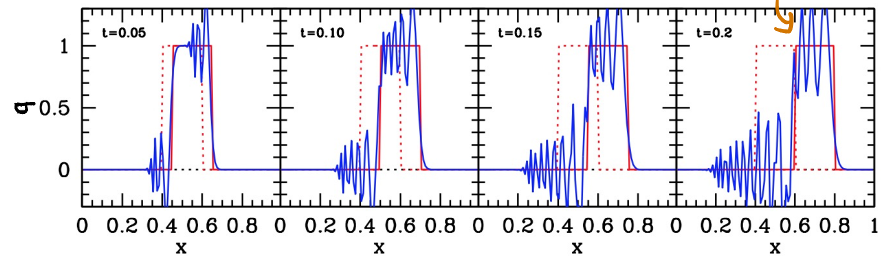
- 100 cells for domain $[0,1] \Rightarrow \Delta x = 0.01$
- $\Delta t = 0.001$
- $v = 1$

$$\alpha_c = \frac{v \Delta t}{\Delta x} = 0.1$$

growing oscillations

Forward-Time-Central-Space (FTCS):

$$q_i^{n+1} = q_i^n - v \frac{\Delta t}{2 \Delta x} \left(q_{i+1}^n - q_{i-1}^n \right)$$

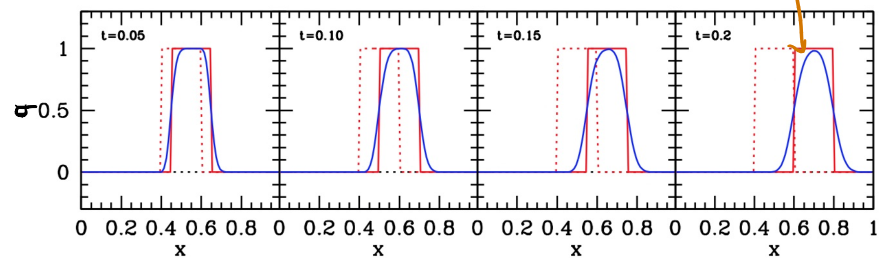


diffusion like

Forward-Time-Backward-Space (FTBS):

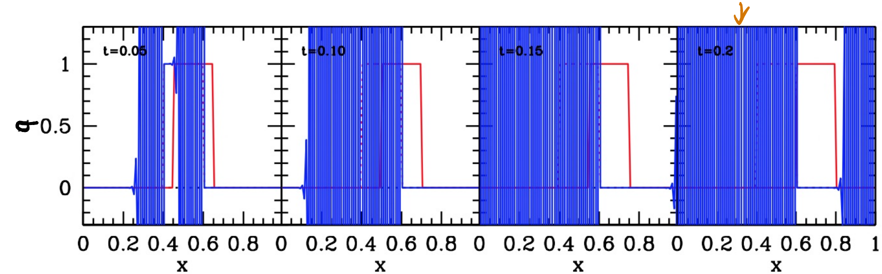
$$q_i^{n+1} = q_i^n - v \frac{\Delta t}{\Delta x} \left(q_i^n - q_{i-1}^n \right)$$

"upwind scheme"



Forward-Time-Forward-Space (FTFS):

$$q_i^{n+1} = q_i^n - v \frac{\Delta t}{\Delta x} \left(q_{i+1}^n - q_i^n \right)$$

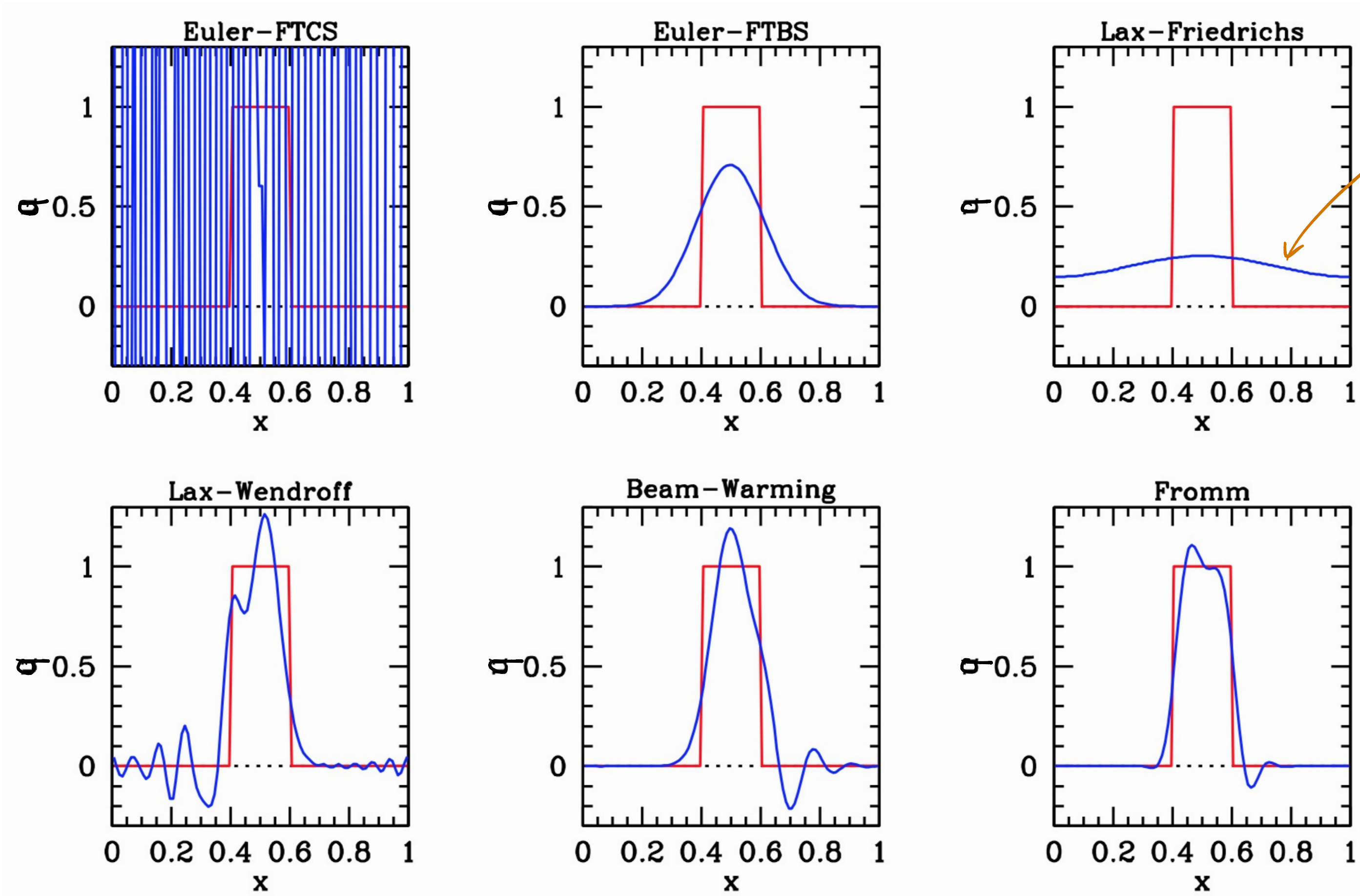


Alternative integration schemes

Euler FTCS	$q_i^{n+1} = q_i^n - v\frac{\Delta t}{2\Delta x}(q_{i+1}^n - q_{i-1}^n)$
Euler FTBS	$q_i^{n+1} = q_i^n - v\frac{\Delta t}{\Delta x}(q_i^n - q_{i-1}^n)$
Euler FTFS	$q_i^{n+1} = q_i^n - v\frac{\Delta t}{\Delta x}(q_{i+1}^n - q_i^n)$
Lax-Friedrichs	$q_i^{n+1} = q_i^n - v\frac{\Delta t}{2\Delta x}(q_{i+1}^n - q_{i-1}^n) + \frac{1}{2}(q_{i+1}^n - 2q_i^n + q_{i-1}^n)$
Lax-Wendroff	$q_i^{n+1} = q_i^n - v\frac{\Delta t}{2\Delta x}(q_{i+1}^n - q_{i-1}^n) + v^2\frac{(\Delta t)^2}{2(\Delta x)^2}(q_{i+1}^n - 2q_i^n + q_{i-1}^n)$
Beam-Warming	$q_i^{n+1} = q_i^n - v\frac{\Delta t}{2\Delta x}(3q_i^n - 4q_{i-1}^n + q_{i-2}^n) + v^2\frac{(\Delta t)^2}{2(\Delta x)^2}(q_{i+1}^n - 2q_{i-1}^n + q_{i-2}^n)$
Fromm	$q_i^{n+1} = q_i^n - v\frac{\Delta t}{4\Delta x}(q_{i+1}^n + 3q_i^n - 5q_{i-1}^n + q_{i-2}^n) + v^2\frac{(\Delta t)^2}{4(\Delta x)^2}(q_{i+1}^n - q_i^n - q_{i-1}^n + q_{i-2}^n)$

Comparison with alternative schemes

t=1.0



numerical viscosity too strong